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Quantum dissipationless random motion

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Abstract : The conditional probability P of quantum dissipationless random motion is derived by incorporating the concepts of wave packet, path integrals, white noise, and velocity distribution. Analytical evolution of P is carried out for the harmonic oscillator as well as free particle cases, and numerical comparison is made with a novel perturbation series for P. The stochastic resonance phenomenon is also studied in a double well. Sharp contrast with the results of quantum Brownian motion is emphasized.

Keywords : Random motion, conditional probability, stochastic resonance

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1. Introduction

While describing the motion of a single particle placed in an environment, one may have to consider, in general, three types of forces *viz.* a conservative force derivable from a mean field, a frictional force proportional to the velocity, and a randomly fluctuating force attributed to noise. To be more precise, consider a one-dimensional problem and let m be the mass, t the time, x the position, v the velocity, $V_R(x)$ the applied or mean field potential, γ the coefficient of friction, $f(t)$ a white Gaussian noise with zero mean, C the autocorrelation strength of the noise, T the absolute temperature of the medium, and $\beta = (kT)^{-1}$ with k being the Boltzmann constant. Depending upon special choices of the empirical parameters γ and C, one can in principle, have the following four different types of single-particle dynamics.

(i) Conservative motion [1] [$\gamma = 0$, $C = 0$) where the deterministic Newtonian trajectories are governed by the shape of the mean field V_R.

(ii) Frictional motion [2] [$\gamma > 0$, $C = 0$] in which the deterministic Rayleigh trajectories are governed by a competition between the mean field and the frictional force.

(iii) Brownian motion [3] [$\gamma > 0$, $C = 2m\gamma kT > 0$] where the Langevin trajectories acquire a probabilistic nature under the simultaneous influence of a frictional force and a random noise which has been generated 'internally' due to thermal fluctuations.

(iv) Dissipationless random motion [4] [$\gamma = 0$, $C > 0$] in which the relevant trajectories again acquire a probabilistic character being subjected to a stochastic force which has been applied 'externally' without any root in thermal fluctuations. In the present paper, we shall focus attention on some deeper properties of this type of motion.

Sizable amount of literature [4] already exists on the motivation, examples, and formulation of *Classical Dissipationless Random Motion* (CDRM) as verified from a brief summary given in items (a)–(d) of Table 1. In particular, Chanana and Menon [4] considered the equation of motion $mdv/dt + \partial V_R/\partial x = f$ which follows from the Lagrangian $L = mv^2/2 - V_R + xf$, assuming a linear coupling to the random force. These authors thereby derived the mechanical trajectories, statistical probabilities, as well as comparison with classical Brownian variances of Chandrasekhar [3a] but the question of extending the whole philosophy to the quantum régime was not attempted.

The aim of the present paper is to formulate the theory of *Quantum Dissipationless Random Motion* (QDRM) subject to the assumptions listed in item (e) of Table 1. In particular, the system variables will be treated quantum-mechanically but the noise will be regarded as classical. The basic question to be addressed in the sequel is the following : 'For a frictionless quantum particle under the joint influence of a conservative potential $V_R(x)$, a white Gaussian external noise $f(t)$, and an input thermal momentum-distribution characterized by temperature T, what is the conditional probability density

$$P \equiv P\left(x_b, t_b \,\middle|\, x_0, 0\right) \tag{1}$$

of finding the particle at the position x_b at time t_b, given that it was initially localized around the point x_0 at the instant $t_a = 0$?' We answer this question below in Section 2 through a sequence of Lemmas A, B, and C. Next, the exact and perturbative formulae for P are compared numerically for the harmonic oscillator well in Section 3. Next, the QDRM formulation is generalized in Section 4 to include the effect of an additional weak sinusoidal modulation, and the existence of the stochastic resonance phenomenon is demonstrated graphically. Finally, Section 5 provides our concluding remarks which also stress the fact that not a single result of the present paper can follow from the standard bath models of quantum Brownian motion (in the small friction limit).

Table 1. Various aspects of classical/quantum dissipationless random motion.

Item	Aspect of CDRM/QDRM
(a)	**Physical picture.** An external stochastic force $f(t)$ has been applied to an otherwise conservatively evolving system.
(b)	**Examples in engineering, chemistry and physics.** If a circuit containing inductance l and capacitance c (is subjected to a random voltage f, then the charge q obeys the analogue equation $ld^2q/dt^2 + q/c = f$. Next, molecular dynamics can be simulated on a computer after producing the "noise" from a random number generator. Again, one may consider the motion of a charged particle in a stochastic magnetic field by using a suitable *velocity-dependent* coupling. Furthermore, one may examine the quantum dynamics of valence-shell protons in the ground state of nuclei exposed to random external gamma-ray pulses. Finally, the charge carriers in a superconducting loop (in which electrical resistance is zero) may be subjected to a stochastic external e.m.f.
(c)	**Theoretical motivations for the study.** Firstly, given a pair of phenomenological parameters γ and C the choice $\gamma = 0$, $C > 0$ is certainly an allowed model which is worth-examining. Secondly, even if γ were nonzero, a canonical transformation $\bar{x} = x\exp(\gamma t/2)$ would reduce the mechanics of the $\bar{x}$ coordinate to an essentially dissipationless form with a modified mean field. Thirdly, just as single-paticle quantum Brownian motion asks what happens to dissipational Kanai [2b] wave functions under internal noise, our proposed QDRM asks what happens to dissipationless Thomas Fermi or Hartree-Fock [1b] wave functions under external noise? Fourthly, since classical motion begins to fail in the presence of penetrable barriers and/or in the region of relatively low temperature, it is obviously desirable to extend the theory of CRDM [4] into the quantum domain. Fifthly, a theory so developed will be applicable to those physical systems which do execute QDRM (cf the examples listed above in item *b*).
(d)	**Basic equation of CDRM [4].** Some important remarks on the equation of motion $m\ddot{x} + \partial V_R/\partial x = f$ are in order We have added a stochastic force straightaway in the Newton-Rayleigh trajectory equation as has been the practice in earlier Langevin-type approaches. A white noise and position-dependent potential have been chosen for the sake of simplicity because inclusion of coloured noise and velocity-dependent mean-fields will make the formalism much more tedious.
(e)	**Semiclassical model of QDRM.** Our path-integral based treatment of the Lagrangian $L = m\dot{x}^2/2 - V_R + xf$ is essentially semiclassical in nature because the particle-coordinate x is quantum-mechanical while the noise f is kept classical. Indeed, the classical correlator $\langle f(t) f(t')\rangle = C\delta(t - t')$ is symmetric in t and t', while the true quantum correlator $\langle \hat{f}(t)\, \hat{f}(t')\rangle$ will not be so for operator-valued noise $\hat{f}(t)$. Moreover, we have adopted a linear coupling term $xf(t)$ because that leads to a position-independent random force which is the simplest thing to have in a starting theory as was first hinted by Feynman [7]. Of course, more general position cum velocity-dependent couplings may have to be used depending on the situation *e.g.* in the presence of a stochastic magnetic field.
(f)	**Brownian movement *versus* dissipationless random motion.** As mentioned in the Introduction, Brownian motion (Characterized by internal thermal noise of correlation strength C^B) is physically different from Dissipationless Random Motion (characterized by externally applied noise of correlation strength C). However, one may still think of a *mathematical* accident in which the Langevin coefficient of friction $\gamma^B \to 0$, $T \to \infty$ such that $C^B = 2m\gamma^B kT \to C$, and one may wonder whether the *algebraic* predictions of the two models coincide. The answer is in the *negative* as was demonstrated classically by Chanana and Menon [4]. The consideration of the quantum case is more intricate and its details are relegated to Table 2.

2. Theory of QDRM

Adopting the notation specified in Section 1, we set up the unperturbed action of our particle as $S^{vx}_{ba} = \int dt\{mv^2/2 - V_R\}$ where the time-integration runs always from $t_a = 0$ to

$t_b > 0$. When a position-independent force $f(t)$ is added, the full Feynman propagator [5] between the points x_a and x_b becomes

$$K_{x_b x_a} = \int Dx \exp\left\{\frac{i}{\hbar}\left(S_{ba}^{vx} + \int dt\, xf\right)\right\}. \tag{2}$$

The input wave function ψ^0 at $t_a = 0$ is taken as a minimum uncertainty packet [6] of width σ_0, mean position x_0, and average momentum p_0, *i.e.*,

$$\psi^0(x_a) = \frac{1}{\left(2\pi\sigma_0^2\right)^{1/4}} \exp\left\{\frac{i}{\hbar} x_a p_0 - \frac{(x_a - x_0)^2}{4\sigma_0^2}\right\}. \tag{3a}$$

The white Gaussian noise [7] has correlation strength C and a probability distributional $\Pi[f]$ given by

$$\left\langle f(t) f(t')\right\rangle = C\delta(t - t'); \quad \Pi[f] = \exp\left\{-(2C)^{-1}\int dt f^2\right\}. \tag{3b}$$

Then the following new Lemmas can be established :

Lemma A. The desired probability (cf. eq. (1)) relevant to QDRM is

$$P = Tr\{\Omega\Lambda\} = \int_{-\infty}^{\infty} dx_a \int_{-\infty}^{\infty} dx'_a \Omega_{x_a x'_a} \Lambda_{x'_a x_a} \tag{4}$$

where

$$\Omega_{x_a x'_a} = (2\pi\sigma_0^2)^{-1/2} \exp\left[-m(x_a - x'_a)^2/(2\hbar^2\beta)\right.$$
$$\left. - (x_a - x_0)^2/(4\sigma_0^2) - (x'_a - x_0)^2/(4\sigma_0^2)\right] \tag{5}$$

$$\Lambda_{x'_a x_a} = \int Dx \int Dx' \exp\left[(i/\hbar)\left\{S_{ba}^{vx} - S_{ba}^{vx'} + (iC/2\hbar)\int dt (x - x')^2\right\}\right]. \tag{6}$$

Here β in eq. (5) denotes $1/kT$ for a reservoir at temperature T, the path $x(t)$ in eq. (6) goes from x_a to x_b, the path $x'(t)$ runs from x'_a to x_b, and the labels (x_b, t_b) have been suppressed.

Proof. The first step in the proof is to take a specific profile of $f(t)$ and to write the pure quantum (*i.e.*, Schrödinger) probability density P_Q of the final state as

$$P_Q = \left|\int_{-\infty}^{\infty} dx_a K_{x_b x_a} \psi^0(x_a)\right|^2. \tag{7}$$

In the second step, we insert the integral representation (2) of $K_{x_b x_a}$ and average this P_Q over the noise distributional $\Pi[f]$ (following an original suggestion due to Feynman [7]) and arrive at the noise-averaged probability

$$P_N = \int_{-\infty}^{\infty} dx'_a \int_{-\infty}^{\infty} dx_a \psi^{0*}(x'_a) \Lambda_{x'_a x_a} \psi^0(x_a). \tag{8}$$

In the third step, a final averaging of this P_N is performed over the Maxwellian momentum distribution $dp_0 \exp\{-\beta p_0^2/2m\}$ leading to the quoted result (4). Note that since x_0 is fixed, an averaging over the full Boltzmannian exp $\{-\beta(p_0^2/2m+V_R(x_0))\}$ is not done here.

Lemma B. For a harmonic oscillator well $V_R(x) = m\omega_R^2 x^2/2$, our P reduces analytically to the normal distribution

$$P = (2\pi\sigma^2)^{-1/2}\exp\left\{-\left(x_b - x_0\cos\theta_b\right)^2/2\sigma^2\right\}, \tag{9}$$

where

$$\theta_b = \omega_R t_b; \quad \sigma^2 = \sigma_c^2 + \sigma_q^2; \tag{10}$$

$$\sigma_c^2 = \frac{\sin^2\theta_b}{m\beta\omega_R^2} + \frac{C(\theta_b - \frac{1}{2}\sin 2\theta_b)}{2m^2\omega_R^3}; \tag{11}$$

$$\sigma_q^2 = \sigma_0^2\cos^2\theta_b + \hbar^2\sin^2\theta_b/(4m^2\omega_R^2\sigma_0^2). \tag{12}$$

Proof. In the harmonic oscillator case, the binary propagator $\Lambda_{x_a'x_a}$ (cf. eq. (6)) is best handled by using the transformation

$$\xi = x - x', \quad \eta = (x+x')/2. \tag{13}$$

After separating out the classical part of the action in the (ξ, η) space we first evaluate the binary path integral $\Lambda_{x_a'x_a}$ using Fourier series. The resulting Gaussian integration over $\xi_a = x_a - x_a'$ and $\eta_a = (x_a + x_a')/2$ in the general formula (4) is then carried out yielding eq (9)

Corollary. For a free particle ($\omega_R = 0$) executing QDRM P is again a normal distribution with mean x_0 and variance σ_f^2 given by

$$\sigma_f^2 = \frac{t_b^2}{m\beta} + \frac{Ct_b^3}{3m^2} + \left(\sigma_0^2 + \frac{\hbar^2 t_b^2}{4m^2\sigma_0^2}\right). \tag{14}$$

The physical reason behind the monotonic growth of σ_f with the time t_b is to be attributed to random noise (with no hindrance from friction).

Lemma C. Let the unperturbed Hamiltonian $H^V = p^2/2m + V_R(x)$ possess discrete set of eigenfunctions $\psi_n(x)$ belonging to the eigenvalues E_n, and suppose the noise is weak in the sense that $C d_0^2 t_b/\hbar^2 \ll 1$ where d_0 is a suitable length scale (*e.g.* the linear size of the ground state). Then, we have a first-order perturbation series neglecting $O(C^2)$ terms

$$P \approx \sum_m\sum_n \Omega_{mn}\{\Lambda_{nm}^V + \Lambda_{nm}^C\}; \tag{15a}$$

$$\Omega_{mn} = \int_{-\infty}^{\infty} dx_a \int_{-\infty}^{\infty} dx_a' \psi_m(x_a)\Omega_{x_a x_a'}\psi_n(x_a'); \tag{15b}$$

$$\Lambda_{nm}^V = \psi_n(x_b)\psi_m(x_b)\cos(\omega_{nm}t_b); \quad \omega_{nm} = (E_n - E_m)/\hbar; \tag{16}$$

$$\Lambda_{nm}^C = \frac{-C}{2\hbar^2}\sum_l\sum_k Z_{lk}^{nm}\left\{\frac{\sin(\omega_{nm}t_b) - \sin(\omega_{lk}t_b)}{\omega_{nm} - \omega_{lk}}\right\}; \tag{17}$$

$$Z_{lk}^{nm} = \{\langle\psi_n|x^2|\psi_l\rangle\delta_{km} + \delta_{nl}\langle\psi_k|x^2|\psi_m\rangle$$
$$- 2\langle\psi_n|x|\psi_l\rangle\langle\psi_k|x|\psi_m\rangle\}\ \psi_l(x_b)\ \psi_k(x_b). \tag{18}$$

Proof. In eq. (6), we use Taylor expansion to expand the exponential factor containing C and use the following quantum mechanical identity valid for any non-negative power s and some chosen time t_1 lying between t_a and t_b :

$$\int Dx \exp\{iS_{ba}^{vx}/\hbar\}\{x(t_1)\}^s$$
$$= \sum_k \sum_m \psi_k(x_b)\langle\psi_k|x^s|\psi_m\rangle\psi_m(x_a)\exp\{-i\mu/\hbar\}, \tag{19}$$

where $\mu = (t_b - t_1)E_k + (t_1 - t_a)E_m$. Employing a similar identity for Dx' path integration in eqs. (4–6) we deduce the result (15).

At this stage, some relevant comments on our formalism are in order. Lemma A (eq. 4) reduces the calculation of P to a quadrature for a general potential well $V(x)$. Lemma B (eqs. (9–12)) for the harmonic oscillator case expresses the net variance σ^2 as the sum of a classical [4] contribution σ_c^2 and a quantum correction σ_q^2 such that the classical probability [4] is achieved if $kT = \beta^{-1} \gg \hbar\omega_R$. Also, P in eq. (9) vanishes if $\sigma_0 \to 0, \infty$ because then σ_q blows up. Next, as $t_b \to \infty$ the variance increases like t_b in eq. (11), and like t_b^3 for a free particle (Corollary eq. 14). Furthermore, Lemma C gives a compact perturbative recipe to evaluate P numerically in the weak noise case provided the unperturbed eigenfunctions ψ_m are known beforehand. Finally, salient differences between our formulation and that of the bath models [3b] will be pointed out later in Section 5 (Table 2).

3. Application of QDRM to the oscillator

We shall compare numerically the exact and perturbative formulae of P in the harmonic oscillator case, setting the units as $\hbar = m = \omega_R = 1$ so that the size of the ground state $d_0 = (\hbar / m\omega_R)^{1/2} = 1$ too. The contribution of noise will be small in the integral $A_{x_a' x_a}$ (cf. eq. (6)) and in the variance σ_c^2 (cf. eq. (11))

$$\text{if} \qquad t_b \ll 1/C; \qquad t_b \ll 1/\beta. \tag{20}$$

For suitable values of the parameters x_0, σ_0, β and C, we began by computing the exact P directly from eq. (9) as a function of t_b and x_b. Next, the perturbation series of eq. (15) was calculated using a set of first 15 eigenfunctions by a procedure due to Sethia *et al* [8], exploiting the concept of the 'short-time propagator'. Our results of P are displayed graphically in Figure 1 for varying times t_b, and in Figure 2 for changing positions x_b. Clearly, the solid and dotted curves agree quite well in the case of high temperature and low noise ($\beta = 0.1$, $C = 0.1$), but differ noticeably in the case of lower temperature and stronger noise ($\beta = 1$, $C = 1$), as expected from the validity/invalidity of the inequality (20).

Table 2. Major differences between quantum dissipationless random motion (QDRM) and quantum brownian motion (QBM) in an ohmic bath at high temperature.

S.No.	Item	QDRM	QBM (table B) [3b]
i.	Lagrangian	It is based on local single-particle Lagrangian $L = mv^2/2 - V_R + xf$, where f is an external stochastic force.	It is based on the nonlocal Feynman-Vernon [10] influence functional for a particle coupled to a bath, so that the noise has internal thermal origin.
ii.	Parameter set	The friction and noise parameters are chosen phenomenologically as $\gamma = 0$, $C > 0$	The parameters are mutually proportional by virtue of the dissipation-fluctuation theorem The weak-friction limit (at large T) says $\gamma^B \to 0$, $C^B \equiv 2m\gamma^B kT \to C$.
iii.	Singularities	The path-integral and semiclassical trajectories are singularity-free at every stage	For the Ohmic spectral density the potential renormalization constant is necessarily divergent even after a careful treatment of initial conditions.
iv.	Physical mechanisms	Fluctuations can go on pumping energy into the system without competing loss	Pumping-in of energy by fluctuations and dissipative loss by friction go on happening continuously
v.	Input wave packet	Our ψ^0 has general mean position x_0 and average momentum p_0.	The packet considered by Caldeira *et al* [3b] is centered at the origin. Explicit calculation of Grabert *et al* are given in the canonical ensemble. The packet of Hu *et al* has no momentum-dependence.
vi.	Noise averaging	In our approach an averaging over noise is done by using the explicit distributional $\Pi[f]$ of eq. [3b].	The distributional $\Pi[f]$ is never explicitly used by quantum bath-model authors. A fluctuation kernel in their effective action arises when the bath coordinates are path-integrated out.
vii.	Maxwellian averaging	We take an initial Maxwellian velocity distribution because, ala Ehrenfest, the input wave packet behaves semiclassically like a particle of position x_0 and momentum p_0. Since x_0 is kept fixed the full Boltzmannian $\exp\{-\beta(p_0^2/2m + V_R(x_0))\}$ is not used.	The Maxwellian input $\langle p_0\rangle = 0$, $\langle p_0^2\rangle = mkT$ was first used by Chandrasekhar [3a] in his classical Langevin-equation based treatment of Brownian motion of the free particle and the harmonic oscillator. Unfortunately, no quantum-bath model worker (except Chanana, Menon and Singh [3b]) has ever used a Maxwellian initial density matrix.

Table 2. (*Cont'd.*).

S.No.	Item	QDRM	QBM (table B) [3b]
viii.	Analytical emphasis	In our treatment of classical as well as quantum dissipationless random motion, almost all results concerning trajectories, probabilities and variances have been derived in explicit form	Complete, closed form answers were derived for classical Brownian motion by Chandrasekhar [3a] However, quantum-bath model results have definite limitations in this respect. For example, the final density matrix by Caldeira *et al* [3b] contains several unevaluated multiple integrals, Grabert *et al* consider explicitly only time-independent variances at equilibrium, and the width formula quoted by Hu *et al* has serious misprints.
ix	Harmonic oscillator variance with quantum correction σ_q^2	In eqs (10–12) we found exactly $\sigma^2 = \sin^2\theta_b / (m\beta\omega_R^2) + C\left(\frac{\sin 2\theta_b}{2}\right) / (2m^2\omega_R^3) + \sigma_q^2$ which can never coincide with the result of QBM.	Suppose we adapt the bath models to incorporate a general input wave packet with Maxwellian averaging Then, in the weak friction limit $\gamma^B t_b \ll 1$, we can prove approximately $\sigma^{B^2} = \sin^2\theta_b / (m\beta\omega_R^2) + C\left(\theta_b \cos^2\theta_b - \frac{\sin 2\theta_b}{2}\right) / \left(2m^2\omega_R^3\right) + \sigma_q^2$
x.	Free particle variance with quantum correction $\sigma_{q,f}^2$	In eq (14) we found exactly $\sigma_f^2 = t_b^2 / m\beta + Ct_b^3 / 3m^2 + \sigma_{q,f}^2$ which can not agree with the QBM result	In the adapted bath model and weak-friction, we can show approximately $\sigma_f^{B^2} = t_b^2 / m\beta - Ct_b^3 / 6m^2 + \sigma_{q,f}^2$
xi.	Expansion for weak noise	Our perturbation series of eq. (15a) calculates the genuine QDRM probability between the configurations $(x_0, 0)$ and (x_b, t_b) in absence of friction in finite temperature surrounding with the concept of wave packet and Maxwellian averaging built-in.	The Feynman-Vernon [10] weak noise expansion calculates only the quantum transition probability from eigenstate $\lvert\phi_n(t_a)\rangle$ to eigenstate $\lvert\phi_m(t_b)\rangle$ in presence of friction in a zero temperature bath without any concept of wave packet or Maxwellian averaging. Hence their series can never coincide with ours
xii.	Stochastic resonance	The theory and illustration of stochastic resonance is straight forward in QDRM framework	A clear-cut study of the stochastic resonance using the full machinery of QBM has not been reported so far

Note that if the t_b axis in Figure 1 is extended further then the exact P will exhibit a damped harmonic behaviour of period $2\pi / \omega_R$.

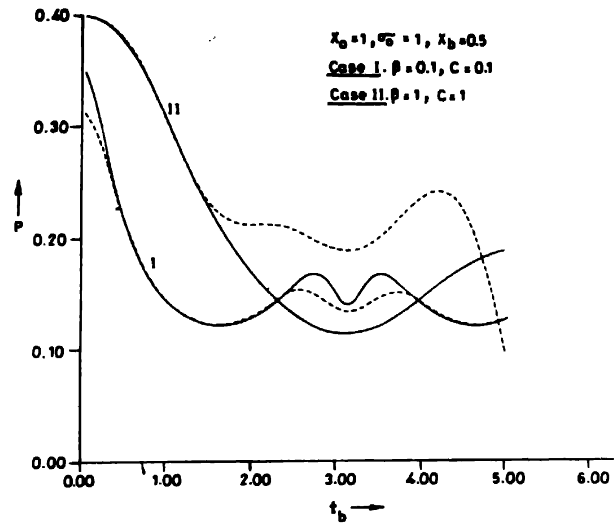

Figure 1. Time-variation of the exact (solid line; eq. 9) and perturbative (dotted line, eq 15) Brownian probabilities for the harmonic oscillator well in the units $\hbar = m = \omega_R = 1$

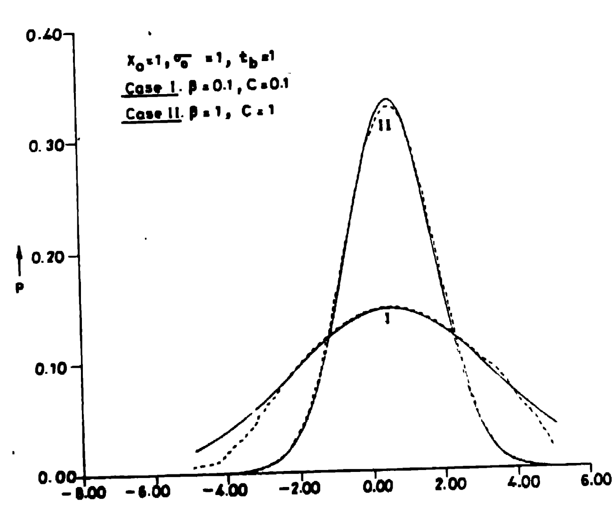

Figure 2. Position-dependence of the exact (solid line) and perturbative (dotted line) probabilities.

4. Application to stochastic resonance

The concept of a stochastic resonance has been introduced previously [9] to describe a curious phenomenon in bistable systems (such as the laser-induced double well) subject to both random and periodic forces; it was found that an increase in the input noise strength can lead to an improvement in the output signal-to-noise ratio. Most of the earlier treatments [9] of this phenomenon were *classical* in nature and applicable to the extremely *overdamped* case; we present below a new treatment which is *quantum* in nature and applicable to the purely *dissipationless* case. We begin by writing the complete propagation kernel as (compare with eq. (2) above)

$$K^r_{x_b x_a} = \int Dx \, \exp\left\{\frac{i}{\hbar}\left[S^{vx}_{ba} + \int dtx(f+y)\right]\right\}, \tag{21}$$

where the superscript r stands for *resonance* and $y = y_{\max} \cos(\omega_s t + \phi)$ is a periodic modulation of amplitude y, angular frequency ω_s, and phase ϕ. Assuming that the noise and modulation are both weak, *i.e.*, $Cd_0^2 t_b / \hbar^2 \ll 1$ and $y_{\max} \ll Cd_0/\hbar$ we can expand the propagator (eq. (21)) in powers of f and y, neglecting $O(f^2)$ and $O(y^2)$ terms but retaining the $O(fy)$ cross term. Retracing the steps analogous to Lemmas A and C above, we arrive at the probability density of the form

$$P^r = \sum_{mn} \Omega_{mn}\left\{\Lambda^V_{nm} + \Lambda^C_{nm} + \Lambda^y_{nm} + \Lambda^{Cy}_{nm}\right\}, \tag{22}$$

where m and n are labels for unperturbed eigenfunctions while the various Λ's are rather complicated, space-time dependent functions on the pattern of eqs. (16)–(18). Clearly, Λ^V_{nm} is the contribution due to the mean field, Λ^C_{nm} that due to noise alone, Λ^y_{nm} that due to pure modulation, and Λ^{Cy}_{nm} that due to noise-modulation *interference*. From eq. (22) we next construct the statistically-averaged trajectory $\underline{x}^r(t_b)$ and its Fourier transform $\underset{\sim}{x}^r(\omega)$ *via*

$$\underline{x}^r(t_b) = \int_{-\infty}^{\infty} dx_b \, x_b \, P^r; \quad \underset{\sim}{x}^r(\omega) = \int_{-\infty}^{\infty} dt_b \, \underline{x}^r(t_b) \, e^{i\omega t_b}. \tag{23}$$

In the same spirit as eq. (22), this Fourier transform will be a sum of four contributions *viz.*

$$\underset{\sim}{x}^r(\omega) = \underset{\sim}{x}^V(\omega) + \underset{\sim}{x}^C(\omega) + \underset{\sim}{x}^y(\omega) + \underset{\sim}{x}^{Cy}(\omega), \tag{24}$$

where $\underset{\sim}{x}^V(\omega) = \sum_{mn} \Omega_{mn} < \psi_n | x | \psi_m > (i\omega)/(\omega^2 - \omega^2_{nm})$ and similarly for the rest provided ω never coincides with any ω_{nm}. Finally, we set up the power spectrum S and the signal-to-noise ratio R (as functions of C for given $y_{\max}$) through

$$S \equiv |\underset{\sim}{x}^r|^2 = S_s + S_b, \qquad R = S_s/S_b, \tag{25}$$

where the *signal* part S_s and the *background* part S_b of the power spectrum are defined by

$$S_s = \left|\underset{\sim}{x}^{Cy}\right|^2 + 2\,\mathrm{Re}\left(\underset{\sim}{x}^{V*} \underset{\sim}{x}^{Cy}\right) + 2\,\mathrm{Re}\left(\underset{\sim}{x}^{C*} \underset{\sim}{x}^{y}\right) + 2\,\mathrm{Re}\left(\underset{\sim}{x}^{C*} \underset{\sim}{x}^{Cy}\right)$$
$$+ 2\,\mathrm{Re}\left(\underset{\sim}{x}^{y*} \underset{\sim}{x}^{Cy}\right), \tag{26}$$

$$S_b = \left|\underline{x}^V\right|^2 + \left|\underline{x}^C\right|^2, \tag{27}$$

where Re stands for the real part, and S_s is adjusted to vanish when either C or y_{max} becomes zero.

Numerical illustration :

In order to illustrate the above theory, we take a symmetric double well potential $V_R = -0.4x^2 + 0.05x^4$ in the $\hbar = m = d_0 = 1$ units. The first 15 eigenvalues and eigenfunctions were generated by using the matrix-diagonalisation code of Sethia *et al* [8]. Keeping β, x_0, σ_0, y_{max}, ϕ, ω_s and ω suitably fixed, we varied the noise strength over the range $0 \leq C \leq 0.1$ and computed the signal-to-noise ratio R from eqs. (25–27) as function of C. The whole procedure was repeated for other choices of the input parameters bearing in mind two precautions *viz.* $\hbar\omega_s$ and $\hbar\omega$ should be much less than the energy-level separation, and the modulation potential amplitude though small compared to the well-depth should be strong enough to cause a transit of the particle from one well to the other.

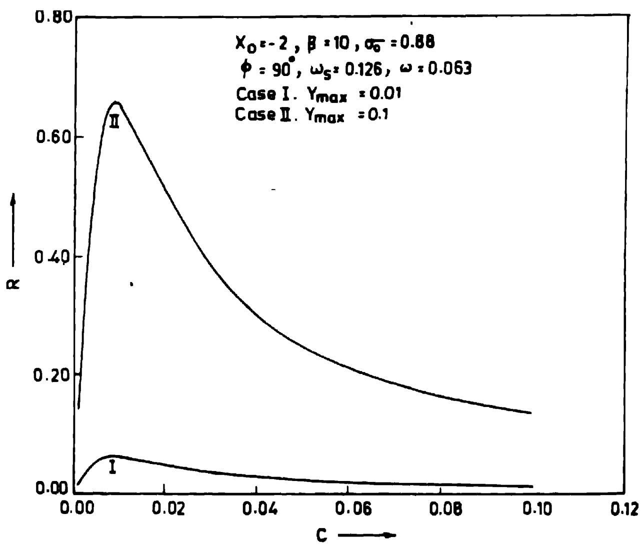

Figure 3. Variation of the signal to noise ratio as a function of the noise strength for different amplitudes of modulation. A symmetric double well potential $V_R = -0.4x^2 + 0.05x^4$ has been taken.

Figure 3 shows our results corresponding to the parameter set

$$\begin{array}{llll} \beta = 10.0, & x_0 = -2, & \sigma_0 = 0.88, & 0.001 \leq y_{max} \leq 0.01 \\ \phi = 90^\circ & \omega_s = 0.126, & \omega = 0.063, & \end{array} \tag{28}$$

assuming that the initial wave packet is centered at the mid-point of the left well. A clear signature of the ***stochastic resonance*** is seen as R increases with C upto a peak value (centered at $C_r = 0.009$ for $y_{max} = 0.001$) and falls-off rapidly as C becomes still higher. Also when y_{max} was changed to 0.01, the location C_r of the peak remained unaltered.

5. Concluding remarks

The material presented in Sections 1–4 above, is entirely *new* and *original* except for those spots where a reference to earlier literature [1–4] was made. Briefly speaking, we have been able to achieve six things in the present paper : (a) the concept of QDRM has been introduced through logical motivations, (b) path-integral based quantum and statistical mechanics of QDRM have been formulated, (c) explicit positional probability densities have been evaluated analytically for the harmonic oscillator and the free particle, (d) a new perturbation formula for weak noise has been developed algebraically and tested numerically, (e) the existence of a dissipationless stochastic resonance in the double well has been demonstrated, and (f) glaring differences with quantum Brownian motion (QBM) have been pinpointed in Table 2.

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